HEAT MAXIMAL FUNCTION
ON A LIE GROUP OF EXPONENTIAL GROWTH

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ABSTRACT. Let G be the Lie group $\mathbb{R}^2 \rtimes \mathbb{R}^+$ endowed with the Riemannian symmetric space structure. Let X_0, X_1, X_2 be a distinguished basis of left-invariant vector fields of the Lie algebra of G and define the Laplacian $\Delta = -(X_0^2 + X_1^2 + X_2^2)$. In this paper, we show that the maximal function associated with the heat kernel of the Laplacian Δ is bounded from the Hardy space H^1 to L^1. We also prove that the heat maximal function does not provide a maximal characterization of the Hardy space H^1.

1. INTRODUCTION

Let G be the Lie group $\mathbb{R}^2 \rtimes \mathbb{R}^+$ where the product rule is the following:

$$(x_1, x_2, a) \cdot (x_1', x_2', a') = (x_1 + a\, x_1', x_2 + a\, x_2', a\, a')$$

for (x_1, x_2, a), $(x_1', x_2', a') \in G$. We shall denote by x the point (x_1, x_2, a). The group G is not unimodular; the right and left Haar measures are given by

$$d\rho(x) = a^{-1}\, dx_1\, dx_2\, da \qquad \text{and} \qquad d\lambda(x) = a^{-3}\, dx_1\, dx_2\, da\,,$$

respectively. The modular function is thus $\delta(x) = a^{-2}$. Throughout this paper, unless explicitly stated, we use the right measure ρ on G and denote by L^p, $\|\cdot\|_p$ and $\langle \cdot, \cdot \rangle$ the L^p-space, the L^p-norm and the L^2-scalar product with respect to the measure ρ.

The group G has a Riemannian symmetric space structure, and the corresponding metric, which we denote by d, is that of the three-dimensional hyperbolic half-space. The metric d is invariant under left translation and given by

$$(1.1) \qquad \cosh r(x) = \frac{a + a^{-1} + a^{-1}|x|^2}{2}\,,$$

where $r(x) = d(x, e)$ denotes the distance of the point x from the identity $e = (0, 0, 1)$ of G and $|x|$ denotes the Euclidean norm of the two-dimensional vector (x_1, x_2), i.e., $|x| = \sqrt{x_1^2 + x_2^2}$. The measure of a hyperbolic ball B_r, centered at the identity and of radius r, behaves like

$$\lambda(B_r) = \rho(B_r) \sim \begin{cases} r^3 & \text{if } r < 1 \\ e^{2r} & \text{if } r \geq 1\,. \end{cases}$$

Thus G is a group of *exponential growth*. In this context, the classical Calderón–Zygmund theory and the classical definition of the atomic Hardy space H^1 (see

[CW1, CW2, St]) do not apply. But W. Hebisch and T. Steger [HS] have constructed a Calderón–Zygmund theory which applies to some spaces of exponential growth, in particular to the space (G, d, ρ) defined above. The main idea is to replace the family of balls which is used in the classical Calderón–Zygmund theory by a suitable family of parallelepipeds which we call *Calderón–Zygmund sets*. The definition appears in [HS] and implicitly in [GS], and reads as follows.

Definition 1.1. A Calderón-Zygmund set is a parallelepiped $R = [x_1 - L/2, x_1 + L/2] \times [x_2 - L/2, x_2 + L/2] \times [ae^{-r}, ae^r]$, where $L > 0$, $r > 0$ and $(x_1, x_2, a) \in G$ are related by

$$e^2 a\, r \leq L < e^8 a\, r \qquad \text{if } r < 1,$$
$$a\, e^{2r} \leq L < a\, e^{8r} \qquad \text{if } r \geq 1.$$

The point (x_1, x_2, a) is called the center of R.

We let $\mathcal{R}$ denote the family of all Calderón–Zygmund sets, and observe that $\mathcal{R}$ is invariant under left translation. Given $R \in \mathcal{R}$, we define its dilated set as $R^* = \{x \in G : d(x, R) < r\}$. There exists an absolute constant $C_0 > 0$ such that $\rho(R^*) \leq C_0\, \rho(R)$ and

$$(1.2) \qquad\qquad R \subset B\big((x_1, x_2, a), C_0 r\big).$$

In [HS] it is proved that every integrable function on G admits a Calderón–Zygmund decomposition involving the family $\mathcal{R}$, and that a new Calderón–Zygmund theory can be developed in this context. By using the Calderón–Zygmund sets, it is natural to introduce an atomic Hardy space H^1 on the group G, as follows (see [V] for details).

We define an *atom* as a function A in L^1 such that

(i) A is supported in a Calderón–Zygmund set R;
(ii) $\|A\|_\infty \leq \rho(R)^{-1}$;
(iii) $\int A\, d\rho = 0$.

The atomic Hardy space is now defined in a standard way.

Definition 1.2. The Hardy space H^1 is the space of all functions f in L^1 which can be written as $f = \sum_j \lambda_j A_j$, where A_j are atoms and λ_j are complex numbers such that $\sum_j |\lambda_j| < \infty$. We denote by $\|f\|_{H^1}$ the infimum of $\sum_j |\lambda_j|$ over such decompositions.

The Calderón–Zygmund theory from [HS] has turned out useful for the study of the boundedness of singular integral operators related to a distinguished Laplacian on G, which is defined as follows.

Let X_0, X_1, X_2 denote the left-invariant vector fields

$$X_0 = a\, \partial_a \qquad X_1 = a\, \partial_{x_1} \qquad X_2 = a\, \partial_{x_2},$$

which span the Lie algebra of G. The Laplacian $\Delta = -(X_0^2 + X_1^2 + X_2^2)$ is a left-invariant operator which is essentially selfadjoint on $L^2(\rho)$. It is well known that the heat semigroup $\big(e^{-t\Delta}\big)_{t>0}$ is given by a kernel p_t, in the sense that $e^{-t\Delta} f = f * p_t$ for suitable functions f. Let $\mathcal{M}$ denote the corresponding maximal operator, defined by

$$(1.3) \qquad\qquad \mathcal{M}f(x) = \sup_{t>0} |f * p_t(x)| \qquad \forall x \in G.$$

M. Cowling, G. Gaudry, S. Giulini and G. Mauceri [CGGM] proved that $\mathcal{M}$ is of weak type $(1,1)$ and bounded on L^p for all $p > 1$. In this paper, we prove the following result.

Theorem 1.3.   (i) *If $f \in H^1$, then $\mathcal{M}f \in L^1$ and*

$$\|\mathcal{M}f\|_1 \le C \|f\|_{H^1},$$

   where C is independent of f.
  (ii) *However, the converse inequality is false, and $f \in L^1$ and $\mathcal{M}f \in L^1$ does not imply $f \in H^1$.*

Remark 1.4. A similar maximal operator $\widetilde{\mathcal{M}}$ can be defined by means of the Poisson semigroup $\left(e^{-t\sqrt{\Delta}}\right)_{t>0}$. Theorem 1.3 is valid also for this maximal operator, since the proof given below remains valid with only small modifications.

In our setting, there is thus no characterization of the atomic H^1 space by means of the heat or the Poisson maximal operator. This is in contrast with the classical theory and other settings, as we now recall.

In the Euclidean case, the atomic Hardy space $H^1(\mathbb{R}^n)$ has a maximal characterization (see [St, Chapter 3]). More precisely, $H^1(\mathbb{R}^n)$ coincides with the set of functions $f \in L^1(\mathbb{R}^n)$ such that $\mathcal{M}_\phi f \in L^1(\mathbb{R}^n)$, where

$$\mathcal{M}_\phi f(x) = \sup_{t>0} \left| f * \phi_t(x) \right| \qquad \forall x \in \mathbb{R}^n,$$

ϕ is any Schwartz function with $\int \phi(x)dx \ne 0$ and $\phi_t(x) = t^{-n}\phi(t^{-1}x)$. Moreover, the norms $\|f\|_{H^1(\mathbb{R}^n)}$ and $\|\mathcal{M}_\phi f\|_{L^1(\mathbb{R}^n)}$ are equivalent.

On a space $\mathbb{X}$ of homogeneous type R.R. Coifman and G. Weiss [CW1, CW2] introduced an atomic Hardy space $H^1_{\mathrm{at}}(\mathbb{X})$. Under some additional assumption, different maximal characterizations of the Hardy space $H^1_{\mathrm{at}}(\mathbb{X})$ were obtained by R.A. Macías and C. Segovia [MS], W.M. Li [L], L. Grafakos, L. Liu and D. Yang [GLY1, GLY2].

On the Euclidean space $\mathbb{R}^n$ endowed with a nondoubling Radon measure μ of polynomial growth, X. Tolsa [T] defined an atomic Hardy space $H^1(\mu)$ and proved that it can be characterized by a grand maximal operator as in the classical setting.

In the setting of a gaussian measure and the Ornstein-Uhlenbeck operator in $\mathbb{R}^n$, G. Mauceri and S. Meda [MM] introduced an atomic H^1 space. For $n = 1$ Mauceri, Meda and Sjögren [MMS] gave a maximal characterization of this space.

In this paper, C denotes a positive, finite constant which may vary from line to line and may depend on parameters according to the context. Given two positive quantities f and g, by $f \lesssim g$ we mean that there exists a constant C such that $f \le C\,g$, and $f \sim g$ means that $g \lesssim f \lesssim g$.

2. The heat maximal function

We shall use the following integration formula (see for instance [CGHM, Lemma 1.3]): for any radial function f such that $\delta^{1/2}f$ is integrable

$$(2.1) \qquad \int_G \delta^{1/2}f \, \mathrm{d}\rho = \int_0^\infty f(r)\, r\, \sinh r \, \mathrm{d}r.$$

The convolution in G is defined by

$$(2.2) \qquad f * g(x) = \int f(xy^{-1})g(y)\,d\rho(y)\,.$$

Let p_t denote the heat kernel associated with Δ. It is well known [CGGM, Theorem 5.3, Proposition 5.4], [ADY, formula (5.7)] that

$$(2.3) \qquad p_t(x) = \frac{1}{8\pi^{3/2}}\,\delta^{1/2}(x)\,\frac{r(x)}{\sinh r(x)}\,t^{-3/2}\,e^{-\frac{r^2(x)}{4t}} \qquad \forall x \in G\,.$$

Note that $p_t = \delta^{1/2}\,h_t$, where h_t is the heat kernel associated with the operator $\mathcal{L} - I$ and $\mathcal{L}$ is the Laplace–Beltrami operator on the three-dimensional hyperbolic space. The kernel h_t and its gradient were studied in [ADY, CGHM].

We remark that given (2.3) it is easy to write the formula for the Poisson kernel. It is enough to replace the factors $t^{-3/2}\,e^{-\frac{r^2(x)}{4t}}$ by $t/(t^2 + r^2(x))^2$ and the constant factor by $1/\pi^2$.

Since $r(x^{-1}) = r(x)$ we have that

$$(2.4) \qquad p_t(x) = \delta(x)\,p_t(x^{-1}) \qquad \forall x \in G\,.$$

Via a change of variables in (2.2), this implies

$$(2.5) \qquad \begin{aligned} f * p_t(x) &= \int f(z^{-1})\,p_t(zx)\,d\rho(z) \\ &= \int f(y)\,p_t(y^{-1}x)\delta(y)\,d\rho(y) \\ &= \delta(x)\int f(y)\,p_t(x^{-1}y)\,d\rho(y)\,. \end{aligned}$$

We shall need the observation that for any $r > 0$

$$(2.6) \qquad \sup_{t>0} t^{-3/2}\,e^{-\frac{r^2}{4t}} \sim r^{-3}\,.$$

We now give some properties of the heat kernel which will be useful in the rest of the paper.

Lemma 2.1. *For any point $x \in G$ the derivatives of the heat kernel p_t along the vector fields X_i, $i = 0, 1, 2$, are the following:*

 (i) $X_i p_t(x) = p_t(x)\frac{x_i}{\sinh r}\left(\frac{1}{r} - \frac{\cosh r}{\sinh r} - \frac{r}{2t}\right)$ *for $i = 1, 2$;*

 (ii) $X_0 p_t(x) = p_t(x)\left(\frac{a}{r\sinh r} - \frac{\cosh r}{r\sinh r} - \frac{a\cosh r}{\sinh^2 r} + \frac{1}{\sinh^2 r} + \frac{r}{2t}\frac{\cosh r - a}{\sinh r}\right),$

where r denotes $r(x)$. Moreover,

 (iii) $\sup_{t>0}|X_i p_t(x)| \lesssim r^{-4}$ *for $i = 0, 1, 2$ and for all $x \in B_1$;*

 (iv) $\sup_{t>0}|X_i p_t(x)| \lesssim \frac{|x|}{a\,r^2\cosh^2 r}$ *for $i = 1, 2$ and for all $x \in B_1^c$;*

 (v) $\sup_{t>0}|X_0 p_t(x)| \lesssim \frac{1}{a\,r^3\cosh r} + \frac{1}{r^2\cosh^2 r}$ *for all $x \in B_1^c$.*

Proof. Parts (i) and (ii) follow by direct computation from [SV, Lemma 2.1].

The proof of (iii) relies on the fact that $p_t = \delta^{1/2}\,h_t$ which implies that for $i = 0, 1, 2$, $|X_i p_t| \lesssim \delta^{1/2}\,(|h_t| + |X_i h_t|)$. From the estimates of $|h_t|$ and its gradient proved in [ADY, Theorem 5.9, Corollary 5.49], (iii) follows.

To prove (iv) and (v), it suffices to combine (i), (ii), (2.3) and (2.6) with $r = r(x) > 1$. $\qquad\square$

For suitable functions F, we denote by $\|\nabla F\|$ the Euclidean norm of the vector $\nabla F = (X_0 F,\, X_1 F,\, X_2 F)$.

By applying the previous lemma, we shall estimate the integral over the complement of a small ball B_{r_1} of the function

$$G_{r_1}(x) = \sup_{B(x, r_1/2)} \sup_{t>0} \|\nabla p_t\|.$$

Lemma 2.2. *For any $r_1 \in (0, 2C_0)$*

$$\int_{B_{r_1}^c} G_{r_1} \, \mathrm{d}\rho \leq \frac{C}{r_1},$$

where C_0 is the constant which appears in (1.2).

Proof. We first derive some pointwise estimates for G_{r_1}, closely similar to those of Lemma 2.1 (iii), (iv) and (v). Let $x = (x_1, x_2, a) \in B_{r_1}^c$. Any point in $B(x, r_1/2)$ can be written $xy = (x_1 + ay_1,\, x_2 + ay_2,\, ab)$, where $y = (y_1, y_2, b) \in B_{r_1/2}$ so that $|y| \lesssim 1$ and $b \sim 1$. The triangle inequality then implies $|r(xy) - r(x)| \leq r_1/2 \leq r(x)/2$ and thus $r(xy) \sim r(x)$ and $\cosh r(xy) \sim \cosh r(x)$. This allows us to conclude from Lemma 2.1 (iii) that

(III) $$G_{r_1}(x) \lesssim r^{-4}, \qquad x \in B_1 \setminus B_{r_1},$$

with $r = r(x)$. Next, we observe that $|x + ay| \lesssim |x| + a$ and see that (iv) implies

(IV) $$\sup_{B(x, r_1/2)} \sup_{t>0} \|X_i p_t\| \lesssim \frac{1 + a^{-1}|x|}{r^2 \cosh^2 r}, \qquad i = 1, 2.$$

Moreover, (v) implies

(V) $$\sup_{B(x, r_1/2)} \sup_{t>0} \|X_0 p_t\| \lesssim \frac{1}{a\, r^3 \cosh r} + \frac{1}{r^2 \cosh^2 r}.$$

Using these three estimates, we shall integrate G_{r_1} in $B_{r_1}^c$. From (III) we get

(2.7) $$\int_{B_1 \setminus B_{r_1}} G_{r_1} \, \mathrm{d}\rho \lesssim \int_{B_1 \setminus B_{r_1}} r(x)^{-4} \, \mathrm{d}\rho(x) \lesssim \frac{1}{r_1},$$

since $\mathrm{d}\rho$ is equivalent to the 3-dimensional Lebesgue measure in B_1.

It remains to show that the integral over B_1^c is finite. To do so, we split B_1^c into two regions, $D_1 = \{x \in B_1^c : |x|^2 > a^2 + 1\}$ and $D_2 = B_1^c \setminus D_1$. If $x \in D_1$, then (1.1) implies that $\cosh r(x) \sim a^{-1}|x|^2$ and $r(x) \sim \log\left(a^{-1}|x|^2\right)$. For $x \in D_2$, one has $\cosh r(x) \sim a + a^{-1}$ and $r(x) \sim \log\left(a + a^{-1}\right)$.

For $i = 1, 2$, we apply (IV) to deal with $X_i p_t$. In D_1 one has $1 \leq a^{-1}|x|$, so that

$$\int_{D_1} \sup_{B(x, r_1/2)} \sup_{t>0} |X_i p_t| \, \mathrm{d}\rho(x)$$

$$\lesssim \int_0^{+\infty} \frac{\mathrm{d}a}{a} \int_{|x| > \sqrt{a^2 + 1}} \frac{a^{-1}|x|}{\left(\log(a^{-1}|x|^2)\right)^2 a^{-2}|x|^4} \, \mathrm{d}x_1 \, \mathrm{d}x_2.$$

Making the change of variables $(x_1, x_2) = \sqrt{a}\,(v_1, v_2)$, we get

$$\int_0^{+\infty} \mathrm{d}a \int_{|v|>\sqrt{a+a^{-1}}} \frac{1}{\left(\log|v|^2\right)^2 \sqrt{a}\,|v|^3}\,\mathrm{d}v_1\,\mathrm{d}v_2$$

(2.8)
$$\lesssim \int_0^{+\infty} \frac{1}{\sqrt{a^2+1}\left(\log(a+a^{-1})\right)^2}\,\mathrm{d}a$$

$$< \infty\,.$$

For the region D_2, we get

$$\int_{D_2} \sup_{B(x,r_1/2)} \sup_{t>0} |X_i p_t|\,\mathrm{d}\rho(x)$$

(2.9)
$$\lesssim \int_0^{+\infty} \frac{\mathrm{d}a}{a} \int_{|x|<\sqrt{a^2+1}} \frac{1+a^{-1}|x|}{\left(\log(a+a^{-1})\right)^2 (a+a^{-1})^2}\,\mathrm{d}x_1\,\mathrm{d}x_2$$

$$\lesssim \int_0^{+\infty} \frac{1}{\left(\log(a+a^{-1})\right)^2 (a+a^{-1})^2}\,(a^2+1)^{3/2}\,\frac{\mathrm{d}a}{a^2}$$

$$< \infty\,.$$

To estimate the integrals involving $X_0 p_t$, we apply (V). The integral over B_1^c of the first term to the right in (V) can be computed by means of (2.1) as follows:

(2.10) $\displaystyle \int_{B_1^c} \frac{1}{a\,r(x_1,x_2,a)^3\,\cosh r(x_1,x_2,a)}\,\mathrm{d}\rho(x_1,x_2,a) = \int_1^\infty \frac{1}{r^3\,\cosh r}\,r\,\sinh r\mathrm{d}r$

$$\lesssim 1\,.$$

The last term in (V) is no larger than the right-hand side of (IV), so its integral is finite by the above.

The estimates (2.7), (2.8), (2.9) and (2.10) give the desired conclusion. $\square$

3. PROOF OF PART (i) OF THEOREM 1.3

We first show that $\mathcal{M}$ is uniformly bounded on atoms.

Proposition 3.1. *There exists a positive constant κ such that for any atom A*

(3.1) $\|\mathcal{M}A\|_1 \le \kappa\,.$

Proof. We first notice that it is sufficient to prove the inequality (3.1) for atoms supported in Calderón–Zygmund sets centred at the identity. Indeed, if B is an atom supported in a Calderón–Zygmund set R centred at a point c_R, then the function A defined by $A(x) = \delta(c_R)^{-1} B(c_R x)$ is an atom supported in the set $c_R^{-1}\cdot R$ which is centred at the identity. Moreover, it is easy to check that $\|\mathcal{M}B\|_1 = \|\mathcal{M}A\|_1$.

Let us thus suppose that A is an atom supported in the Calderón–Zygmund set $R = Q \times [\mathrm{e}^{-r_0}, \mathrm{e}^{r_0}] = [-L/2, L/2] \times [-L/2, L/2] \times [\mathrm{e}^{-r_0}, \mathrm{e}^{r_0}]$. We study the cases when $r_0 < 1$ and $r_0 \ge 1$.

Case $r_0 < 1$. By (1.2) $R \subset B_{C_0 r_0}$. Since $r_0 < 1$, the set $\tilde{R} = B_{2C_0 r_0}$ has measure comparable with $\rho(R)$, and we get
(3.2)
$$\|\mathcal{M}A\|_{L^1(\tilde{R})} \le \rho(\tilde{R})^{1/2} \|\mathcal{M}\|_{L^2 \to L^2} \|A\|_2 \lesssim \rho(R)^{1/2} \|\mathcal{M}\|_{L^2 \to L^2} \rho(R)^{-1/2} \lesssim 1\,.$$

It remains to consider $\mathcal{M}A$ on the complement of $\tilde{R}$. For any point x in $\tilde{R}^c$, we obtain from (2.5) and the cancellation condition of the atom

$$(3.3) \qquad A * p_t(x) = \delta(x) \int_R A(y) \left[p_t(x^{-1}y) - p_t(x^{-1}) \right] d\rho(y).$$

Let y be any point in the ball $B(e, C_0 r_0)$. From the Mean Value Theorem in $\mathbb{R}^3$, we obtain

$$|p_t(y) - p_t(e)| \lesssim r_0 \sup_{B(e, C_0 r_0)} \|\nabla p_t\|.$$

Replacing here p_t by its translate $p_t(x^{-1}\cdot)$ and using the left invariance of ∇, we conclude

$$|p_t(x^{-1}y) - p_t(x^{-1})| \lesssim r_0 \sup_{B(x^{-1}, C_0 r_0)} \|\nabla p_t\|.$$

Since $\|A\|_1 \le 1$, we conclude from this and (3.3) that

$$|A * p_t(x)| \lesssim r_0 \delta(x) \sup_{B(x^{-1}, C_0 r_0)} \|\nabla p_t\|,$$

and so

$$|\mathcal{M}A(x)| \lesssim r_0 \delta(x) \sup_{B(x^{-1}, C_0 r_0)} \sup_{t>0} \|\nabla p_t\|.$$

We now integrate over $x \in \tilde{R}^c$, inverting the variable x. Then Lemma 2.2 with $r_1 = 2 C_0 r_0$ leads to

$$\int_{\tilde{R}^c} |\mathcal{M}A(x)| \, d\rho(x) \lesssim 1,$$

which ends the case $r_0 < 1$.

Case $r_0 \ge 1$. In this case we define $R^{**} = \{x : |x| < \gamma L, \frac{1}{L} < a < \gamma L\}$, where $\gamma > 1$ is a suitable constant. Then $\rho(R^{**})$ is comparable to $\rho(R)$, and there exists a constant C such that
(3.4)
$$\|\mathcal{M}A\|_{L^1(R^{**})} \le \rho(R^{**})^{1/2} \|\mathcal{M}\|_{L^2 \to L^2} \|A\|_2 \le C \, \rho(R)^{1/2} \|\mathcal{M}\|_{L^2 \to L^2} \, \rho(R)^{-1/2} \le C \,.$$

It remains to consider $\mathcal{M}A$ on the complementary set of R^{**}. We write $(R^{**})^c = \Omega_1 \cup \Omega_2 \cup \Omega_3$, where

$$\Omega_1 = \left\{ x \in G : a < \frac{1}{L}, |x| < \gamma L \right\},$$
$$\Omega_2 = \left\{ x \in G : a > \gamma L, |x| < a \right\},$$
$$\Omega_3 = \left\{ x \in G : |x| > \max(a, \gamma L) \right\}.$$

In (3.3) we write $y = (y_1, y_2, b) \in R$ and $x = (x_1, x_2, a)$ so that $x^{-1}y = \left(a^{-1}(y_1 - x_1), a^{-1}(y_2 - x_2), a^{-1}b \right)$. We now study $\mathcal{M}A$ on $\Omega_1, \Omega_2, \Omega_3$ separately.

For $x \in \Omega_1$ and $y \in R$, (1.1) implies

$$\cosh r(x^{-1}y) \ge \frac{a^{-1}b + a^{-1}b^{-1}|x - y|^2}{2} \ge \frac{a^{-1}b}{2} \ge \frac{a^{-1/2}}{2} \ge \frac{e}{2},$$

the last two inequalities since $b \geq e^{-r_0} > L^{-1/2} > a^{1/2}$ and $a^{-1/2} > e^{r_0}$, by the definitions of Ω_1 and Calderón–Zygmund sets. Therefore, by (2.6)

$$\sup_{t>0} p_t(x^{-1}y) \lesssim \delta^{1/2}(x^{-1}y)\, r(x^{-1}y)^{-3}\, \frac{r(x^{-1}y)}{\sinh r(x^{-1}y)}$$

$$\lesssim (a^{-1}b)^{-1}\, \frac{(\log a^{-1})^{-2}}{a^{-1}b^{-1}(b^2 + |x-y|^2)}$$

$$= \frac{a^2}{(\log a^{-1})^2(b^2 + |x-y|^2)}\,.$$

This implies that for all x in Ω_1,

$$\sup_{t>0}|A * p_t(x)| = \delta(x)\sup_{t>0}\left|\int_R A(y)\, p_t(x^{-1}y)\, d\rho(y)\right|$$

$$\lesssim \delta(x)\,\|A\|_\infty \int_R \sup_{t>0} p_t(x^{-1}y)\, d\rho(y)$$

$$\lesssim \delta(x)\,\rho(R)^{-1}\frac{a^2}{(\log a^{-1})^2} \times$$

$$\times \int_{e^{-r_0}}^{e^{r_0}} \int_{|y_1|<L/2}\int_{|y_2|<L/2} \frac{1}{b^2+|x-y|^2}\, dy_1\, dy_2\, \frac{db}{b}$$

$$\lesssim a^{-2}\,(L^2 r_0)^{-1}\frac{a^2}{(\log a^{-1})^2}\int_{e^{-r_0}}^{e^{r_0}} \log\left(\frac{L}{b}\right)\frac{db}{b}$$

$$\lesssim \frac{r_0}{L^2(\log a^{-1})^2}\,.$$

By taking the integral over the region Ω_1, we obtain that

$$(3.5)\qquad \int_{\Omega_1}\mathcal{M}A(x)\, d\rho(x) \lesssim \frac{r_0}{L^2}\int_0^{1/L}\frac{da}{a\,(\log a^{-1})^2}\int_{|x|<\gamma L} dx_1\, dx_2$$

$$\lesssim \frac{r_0}{\log L}\sim 1\,.$$

In order to study the integral of $\mathcal{M}A$ on the remaining regions, we use the cancellation condition of the atom and write

$$A * p_t(x) = \delta(x)\int_R A(y)\left[p_t(x^{-1}y) - p_t(x^{-1}\tilde{y}) + p_t(x^{-1}\tilde{y}) - p_t(x^{-1})\right]d\rho(y)\,,$$

where for $y = (y_1, y_2, b) \in R$ we write $\tilde{y} = (y_1, y_2, 1)$ and thus $y = \tilde{y}\exp(\log b\, X_0)$. Setting $q_1(x) = \sup_{t>0,\, y\in R}|p_t(x^{-1}y) - p_t(x^{-1}\tilde{y})|$ and $q_2(x) = \sup_{t>0,\, y\in R}|p_t(x^{-1}\tilde{y}) - p_t(x^{-1})|$, we conclude that

$$\mathcal{M}A(x) \leq \delta(x)\left(q_1(x) + q_2(x)\right)\int_R |A(y)|\, d\rho(y) \leq \delta(x)\left(q_1(x) + q_2(x)\right).$$

We first estimate q_2. Observe that $\tilde{y} = \exp(y_1 X_1 + y_2 X_2)$, so that

$$p_t(x^{-1}\tilde{y}) - p_t(x^{-1}) = \int_0^1 (y_1 X_1 + y_2 X_2)p_t\big(x^{-1}\exp(sy_1 X_1 + sy_2 X_2)\big)\, ds\,.$$

Here $x^{-1}\exp(sy_1 X_1 + sy_2 X_2) = \big(a^{-1}(sy_1 - x_1), a^{-1}(sy_2 - x_2), a^{-1}\big)$ and

$$2\cosh r\big(x^{-1}\exp(sy_1 X_1 + sy_2 X_2)\big) = a^{-1} + a + a\, a^{-2}|sy - x|^2\,,$$

because of (1.1). We consider $0 < s < 1$ and $y \in R$, which implies that $|y_1|, |y_2| \leq L/2$.

For $x \in \Omega_2$ we get $|sy - x| \lesssim a$, $\cosh r\big(x^{-1}\exp(sy_1 X_1 + sy_2 X_2)\big) \sim a$ and also $r\big(x^{-1}\exp(sy_1 X_1 + sy_2 X_2)\big) \sim \log a$, so that Lemma 2.1 (iv) implies that

$$\big|(y_1 X_1 + y_2 X_2)p_t\big(x^{-1}\exp(sy_1 X_1 + sy_2 X_2)\big)\big| \lesssim L\frac{a\,a^{-1}\,a}{a^2(\log a)^2} = \frac{L}{a\,(\log a)^2}\,.$$

Thus

$$
\begin{aligned}
\int_{\Omega_2} \delta(x)\, q_2(x)\, \mathrm{d}\rho(x) &\lesssim L \int_{\Omega_2} \frac{1}{a^3(\log a)^2}\, \mathrm{d}\rho(x) \\
&\lesssim L \int_{\gamma L}^{\infty} \frac{1}{a^3(\log a)^2} \int_{|x|<a} \mathrm{d}x_1\, \mathrm{d}x_2\, \frac{\mathrm{d}a}{a} \\
&\lesssim L \int_{\gamma L}^{\infty} \frac{\mathrm{d}a}{a^2(\log a)^2} \\
&\lesssim \frac{1}{(\log L)^2}\,.
\end{aligned}
$$

(3.6)

For $x \in \Omega_3$ we have $|x| > \gamma L$, so that $|sy - x| \sim |x| \geq a$, and it follows that $\cosh r\big(x^{-1}\exp(sy_1 X_1 + sy_2 X_2)\big) \sim a^{-1}|x|^2$. Then Lemma 2.1 (iv) implies

$$\big|(y_1 X_1 + y_2 X_2)p_t\big(x^{-1}\exp(sy_1 X_1 + sy_2 X_2)\big)\big| \lesssim L\frac{a\,a^{-1}\,|x|}{a^{-2}|x|^4\big(\log(a^{-1}|x|^2)\big)^2}\,.$$

Thus

$$
\begin{aligned}
\int_{\Omega_3} \delta(x)\, q_2(x)\, \mathrm{d}\rho(x) &\lesssim L \int_{\Omega_3} \frac{1}{|x|^3\big(\log(a^{-1}|x|^2)\big)^2}\, \mathrm{d}x_1\, \mathrm{d}x_2\, \frac{\mathrm{d}a}{a} \\
&\lesssim L \int_{|x|>\gamma L} \frac{\mathrm{d}x_1\, \mathrm{d}x_2}{|x|^3} \int_{a<|x|} \frac{\mathrm{d}a}{a\big(\log(a^{-1}|x|^2)\big)^2} \\
&\lesssim L \int_{|x|>\gamma L} \frac{\mathrm{d}x_1\, \mathrm{d}x_2}{|x|^3 \log|x|} \\
&\lesssim \frac{1}{\log L}\,.
\end{aligned}
$$

(3.7)

To estimate q_1, we write

$$p_t(x^{-1}y) - p_t(x^{-1}\tilde{y}) = \log b \int_0^1 X_0 p_t\big(x^{-1}\tilde{y}\exp(s\log b\, X_0)\big)\, \mathrm{d}s\,.$$

Here $x^{-1}\tilde{y}\exp(s\log b\, X_0) = \big(a^{-1}(y_1 - x_1), a^{-1}(y_2 - x_2), a^{-1}b^s\big)$, and so

$$2\cosh r\big(x^{-1}\tilde{y}\exp(s\log b\, X_0)\big) = a^{-1}b^s + a\,b^{-s} + a\,b^{-s}a^{-2}|y - x|^2\,.$$

For $x \in \Omega_2$, we get $|y - x| \lesssim a$ and $\cosh r\big(x^{-1}\tilde{y}\exp(s\log b\, X_0)\big) \sim a\,b^{-s}$ which implies $r\big(x^{-1}\tilde{y}\exp(s\log b\, X_0)\big) \sim \log a$ because $a > \gamma L > \gamma \mathrm{e}^{2r_0}$ and $\mathrm{e}^{-r_0} \leq b^{-s} \leq \mathrm{e}^{r_0}$. Then Lemma 2.1 (v) implies that

$$\big|X_0 p_t\big(x^{-1}\tilde{y}\exp(s\log b\, X_0)\big)\big| \lesssim \frac{1}{a^{-1}b^s\,a\,b^{-s}\,(\log a)^3} + \frac{1}{a^2\,b^{-2s}(\log a)^2}$$

$$\lesssim \frac{1}{(\log a)^3}\,.$$

Thus

$$\int_{\Omega_2} \delta(x)\, q_1(x)\, \mathrm{d}\rho(x) \lesssim r_0 \int_{\Omega_2} \frac{1}{a^2(\log a)^3}\, \mathrm{d}\rho(x)$$

(3.8)
$$\lesssim r_0 \int_{a>\gamma L} \frac{1}{a^3(\log a)^3} \int_{|x|<a} \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}a$$

$$\lesssim r_0 \frac{1}{(\log L)^2}$$

$$\lesssim \frac{1}{r_0}\,.$$

For $x \in \Omega_3$ we have $|y-x|^2 \sim |x|^2$ and $\cosh r\big(x^{-1}\tilde{y}\exp(s\log b\, X_0)\big) \sim a\, b^{-s}\, a^{-2}|x|^2$. It follows that $\log(a^{-1}|x|^2) \lesssim r\big(x^{-1}\tilde{y}\exp(s\log b\, X_0)\big)$. Lemma 2.1 (v) now implies

$$\big|X_0 p_t\big(x^{-1}\tilde{y}\exp(s\log b\, X_0)\big)\big| \lesssim \frac{1}{a^{-1}\, b^s\, a^{-1}\, b^{-s}|x|^2(\log(a^{-1}|x|^2)^3} + \frac{1}{a^{-2}\, b^{-2s}|x|^4(\log a^{-1}|x|^2)^2}$$

$$\lesssim \frac{a^2}{|x|^2(\log(a^{-1}|x|^2)^3} + \frac{a^2}{|x|^3(\log(a^{-1}|x|^2)^2}\,;$$

in the last term we have used the inequalities $b^{2s} \le \mathrm{e}^{2r_0} < L < |x|$. Thus

$$\int_{\Omega_3} \delta(x)\, q_1(x)\, \mathrm{d}\rho(x) \le r_0 \int_{|x|>\gamma L} \frac{1}{|x|^2} \int_{a<|x|} \frac{\mathrm{d}a}{a(\log(a^{-1}|x|^2)^3}\, \mathrm{d}x_1\, \mathrm{d}x_2$$

(3.9)
$$+ r_0 \int_{|x|>\gamma L} \frac{1}{|x|^3} \int_{a<|x|} \frac{\mathrm{d}a}{a(\log(a^{-1}|x|^2)^2}\, \mathrm{d}x_1\, \mathrm{d}x_2$$

$$\le r_0 \int_{|x|>\gamma L} \frac{\mathrm{d}x_1\, \mathrm{d}x_2}{|x|^2(\log|x|)^2}\, \mathrm{d}x_1\, \mathrm{d}x_2$$

$$+ r_0 \int_{|x|>\gamma L} \frac{\mathrm{d}x_1\, \mathrm{d}x_2}{|x|^3\, \log|x|}$$

$$\le \frac{r_0}{\log L} + \frac{r_0}{L\log L}$$

$$\lesssim 1\,.$$

Combining the above, we obtain the estimate $\int \mathcal{M}A\, \mathrm{d}\rho \lesssim 1$ in the case $r_0 \ge 1$, which completes the proof of Proposition 3.1. □

We can now prove part (i) of Theorem 1.3.

Proof. Take a function $f \in H^1$, and let us prove that $\mathcal{M}f \in L^1$. There exists a sequence of atoms $\{A_j\}_j$ and complex numbers $\{\lambda_j\}_j$ such that $f = \sum_{j=1}^{\infty} \lambda_j A_j$ and $\sum_{j=1}^{\infty} |\lambda_j| \le 2\|f\|_{H^1}$. By (2.5)

$$f * p_t(x) = \int f(y)\, \delta(x)\, p_t(x^{-1}y)\, \mathrm{d}\rho(y)\,.$$

In view of [CGGM, Theorem 2.1], the heat kernel p_t is for each $t > 0$ a bounded function on G. Thus for each $t > 0$ and $x \in G$ the function $y \mapsto \delta(x)\, p_t(x^{-1}y)$ is in L^∞. But then

$$\int f(y)\delta(x)\, p_t(x^{-1}y)\, \mathrm{d}\rho(y) = \lim_{N\to\infty} \sum_{j=1}^{N} \int \lambda_j\, A_j(y)\, \delta(x)\, p_t(x^{-1}y)\, \mathrm{d}\rho(y)\,,$$

and so

$$|f * p_t(x)| \leq \sum_{j=1}^{\infty} |\lambda_j| \, |A_j * p_t(x)| \, .$$

Taking the supremum in t, we get

$$\mathcal{M}f(x) \leq \sum_{j=1}^{\infty} |\lambda_j| \mathcal{M}A_j(x) \, ,$$

and the result now follows from Proposition 3.1. $\square$

4. Proof of Part (ii) of Theorem 1.3

Even though the two statements in Part (ii) are equivalent, we choose to prove both, by means of simple, explicit examples. First we recall the definition and some properties of the space BMO (see [V] for details).

Definition 4.1. The space $\mathcal{BMO}$ is the space of all functions in L^1_{loc} such that

$$\sup_{R \in \mathcal{R}} \frac{1}{\rho(R)} \int_R |g - g_R| \, \mathrm{d}\rho < \infty \, ,$$

where g_R denotes the mean value of g in the set R, and $\mathcal{R}$ is the family of Calderón–Zygmund sets. The space BMO is $\mathcal{BMO}$ modulo the subspace of constant functions. It is a Banach space endowed with the norm

$$\|g\|_{BMO} = \sup \left\{ \frac{1}{\rho(R)} \int_R |g - g_R| \, \mathrm{d}\rho : \ R \in \mathcal{R} \right\} .$$

The space BMO can be identified with the dual of the Hardy space H^1. More precisely, for any g in BMO the functional ℓ defined on any atom A by

$$\ell(A) = \int g \, A \, \mathrm{d}\rho \, ,$$

extends to a bounded functional on H^1. Furthermore, any functional in the dual of H^1 is of this type, and $\|\ell\|_{(H^1)^*} \sim \|g\|_{BMO}$. Given functions g in BMO and f in H^1 we shall denote by $\langle g, f \rangle$ the action of g on f in this duality.

Aiming at Theorem 1.3 (ii), we shall construct a family of functions $\{f_L\}_{L>2}$ in H^1 such that

$$\lim_{L \to +\infty} \frac{\|f_L\|_{H^1}}{\|\mathcal{M}f_L\|_1} = +\infty \, .$$

Fix $L > 2$ and consider the rectangles $R_0 = [-1, 1] \times [-1, 1] \times [\frac{1}{e}, e]$ and $R_L = (L, 0, 1) \cdot R_0 = [L - 1, L + 1] \times [-1, 1] \times [\frac{1}{e}, e]$. We then define $f_L = \chi_{R_L} - \chi_{R_0}$. Obviously f_L is a multiple of an atom, so it lies in the Hardy space H^1. We shall estimate the H^1-norm of the function f_L from below, by applying the duality between H^1 and BMO.

Lemma 4.2. *There exists a positive constant C such that*

$$\|f_L\|_{H^1} \geq C \log L \qquad \forall L > 2 \, .$$

Proof. We consider the function h in $BMO(\mathbb{R})$ given by $h(s) = \log |s|$ for all s in $\mathbb{R}$ and define $g(x_1, x_2, a) = h(x_1) = \log |x_1|$. First we observe that g is in BMO. Indeed, let $R = Q_1 \times Q_2 \times I$ be a Calderón–Zygmund set, where Q_1, Q_2 and I are

intervals of the form given by Definition 1.1. One finds that g_R coincides with the mean h_{Q_1} of h in Q_1, and so

$$\frac{1}{\rho(R)} \int_R |g - g_R| \, \mathrm{d}\rho = \frac{1}{|Q_1|} \int_{Q_1} |h(x_1) - h_{Q_1}| \, \mathrm{d}x_1 \leq \|h\|_{BMO(\mathbb{R})}\,.$$

This implies that g is in BMO and $\|g\|_{BMO} = \|h\|_{BMO(\mathbb{R})}$.

Since f_L is a multiple of an atom, $\langle g, f_L \rangle = \int g \, f_L \, \mathrm{d}\rho$, and it is easy to verify that $\left| \int g \, f_L \, \mathrm{d}\rho \right| \geq C \log L$. On the other hand, $|\langle g, f_L \rangle| \lesssim \|g\|_{BMO} \|f_L\|_{H^1}$. The lemma follows.

$\square$

Remark 4.3. The estimate of Lemma 4.2 is sharp, which can be seen as follows. With $J = \left[(\log L)/2 \right]$, we shall write the function f_L as a linear combination of $2J + 1 \sim \log L$ atoms. The intuitive idea is to construct a chain of atoms which reach the same "height" $L/2$ in the a-variable as the geodesics connecting the sets R_0 and R_L. More precisely, for $j = 0, 1, ..., J-1$ set

$$P_j = (0, 0, \mathrm{e}^{2j}) \cdot R_0 = (-\mathrm{e}^{2j}, \mathrm{e}^{2j}) \times (-\mathrm{e}^{2j}, \mathrm{e}^{2j}) \times (\mathrm{e}^{2j-1}, \mathrm{e}^{2j+1})\,,$$

and define the functions

$$A_j = \frac{\rho(R_0)}{\rho(P_j)} \chi_{P_j} - \frac{\rho(R_0)}{\rho(P_{j+1})} \chi_{P_{j+1}}\,.$$

Similarly, set

$$Q_j = (L, 0, 1) \cdot P_j = (L - \mathrm{e}^{2j}, L + \mathrm{e}^{2j}) \times (-\mathrm{e}^{2j}, \mathrm{e}^{2j}) \times (\mathrm{e}^{2j-1}, \mathrm{e}^{2j+1})\,,$$

and define the functions

$$B_j = \frac{\rho(R_L)}{\rho(Q_j)} \chi_{Q_j} - \frac{\rho(R_L)}{\rho(Q_{j+1})} \chi_{Q_{j+1}}\,.$$

Notice that $\rho(P_j) = \rho(Q_j) \sim \mathrm{e}^{4j}$ for each j, and it is easy to see that there exist Calderón–Zygmund sets $\tilde{P}_j$ and $\tilde{Q}_j$ such that $\rho(\tilde{P}_j) = \rho(\tilde{Q}_j) \sim e^{4j}$ and

$$\mathrm{supp}(A_j) = P_j \cup P_{j+1} \subset \tilde{P}_j \qquad \text{and} \qquad \mathrm{supp}(B_j) = Q_j \cup Q_{j+1} \subset \tilde{Q}_j\,.$$

Thus, A_j and B_j are multiples of atoms. Moreover, $P_J \cup Q_J$ is contained in a Calderón–Zygmund set whose measure is comparable with $\rho(P_J)$, and thus $\frac{\rho(R_0)}{\rho(P_J)} \left[\chi_{P_J} - \chi_{Q_J} \right]$ is also a multiple of an atom.

We conclude that

$$f_L = \sum_{j=0}^{J-1} B_j - \sum_{j=0}^{J-1} A_j - \frac{\rho(R_0)}{\rho(P_J)} \left[\chi_{P_J} - \chi_{Q_J} \right],$$

so that $\|f_L\|_{H^1} \sim \log L$.

Proposition 4.4. *There exists a positive constant C such that for $L > C$*

$$\|\mathcal{M} f_L\|_1 \lesssim \log \log L\,.$$

Proof. Denote by $2R_0$ the rectangle $[-2, 2] \times [-2, 2] \times [\frac{1}{\mathrm{e}^2}, \mathrm{e}^2]$ and by $2R_L$ the rectangle $(L, 0, 1) \cdot (2R_0)$. We shall estimate the L^1-norm of the maximal function $\mathcal{M} f_L$ by integrating it over different regions of the space.

Step 1. The operator $\mathcal{M}$ is a contraction on L^∞, so that $\mathcal{M}f_L \leq 1$, and since $\rho(2R_0) = \rho(2R_L) \sim 1$, clearly

$$\text{(4.1)} \qquad \int_{2R_0 \cup 2R_L} \mathcal{M}f_L \, d\rho \lesssim 1 \,.$$

Step 2. Choose a ball $B = B(e, r_B)$ with $r_B = (\log L)^\alpha$, where $\alpha > 2$ is a constant. Then (1.1) implies that $B \supset 2R_0 \cup 2R_L$ if L is large enough. We shall estimate the maximal function on $B \setminus (2R_L \cup 2R_0)$. From (2.5) we see that for any x

$$\text{(4.2)} \qquad \mathcal{M}\chi_{R_0}(x) = \sup_t \int \chi_{R_0}(y) \, p_t(y^{-1}x) \, d\lambda(y) \lesssim \sup_{y \in R_0} \sup_t p_t(y^{-1}x) \,.$$

If $x \in (2R_0)^c$ and $y \in R_0$, then $\delta^{1/2}(y^{-1}x) \sim \delta^{1/2}(x)$ and

$$|r(y^{-1}x) - r(x)| = |d(y,x) - d(x,e)| \leq d(y,e) \leq C \,.$$

Applying (2.3) and (2.6), we see that then

$$\sup_{y \in R_0} \sup_t p_t(y^{-1}x) \lesssim \delta^{1/2}(x) \, \frac{1}{r(x)^2 \sinh r(x)} \,.$$

It follows that for any $x \in (2R_0)^c$

$$\text{(4.3)} \qquad \mathcal{M}\chi_{R_0}(x) \lesssim \delta^{1/2}(x) \, \frac{1}{r(x)^2 \, \sinh r(x)} \,.$$

Since $2R_0$ can be seen to contain the unit ball $B(e,1)$, we apply (2.1) and (4.3), getting

$$\text{(4.4)} \qquad \begin{aligned} \int_{B \setminus 2R_0} \mathcal{M}\chi_{R_0} \, d\rho &\lesssim \int_1^{r_B} r^{-2} \, \frac{1}{\sinh r} \, r \, \sinh r \, dr \\ &= \log r_B \\ &\lesssim \log \log L \,. \end{aligned}$$

Observe now that $\mathcal{M}\chi_{R_L}(x) = \mathcal{M}\chi_{R_0}(\tau_L x)$, where $\tau_L x = (-L, 0, 1) \cdot x$, for any x in G. Using the facts that $\tau_L R_L = R_0$ and $\tau_L B \subset B(e, 2r_B) = 2B$ for large L, and changing variable $\tau_L x = v$, we obtain

$$\text{(4.5)} \qquad \begin{aligned} \int_{B \setminus 2R_L} \mathcal{M}\chi_{R_L}(x) \, d\rho(x) &= \int_{B \setminus 2R_L} \mathcal{M}\chi_{R_0}(\tau_L x) \, d\rho(x) \\ &= \int_{\tau_L B \setminus \tau_L(2R_L)} \mathcal{M}\chi_{R_0}(v) \, \delta(-L, 0, 1) \, d\rho(v) \\ &\leq \int_{2B \setminus 2R_0} \mathcal{M}\chi_{R_0}(v) \, d\rho(v) \\ &\lesssim \log(2r_B) \\ &\lesssim \log \log L \,, \end{aligned}$$

as before.

Thus, from (4.4) and (4.5) we deduce that

$$\text{(4.6)} \qquad \int_{B \setminus (2R_0 \cup 2R_L)} \mathcal{M}f_L \, d\rho \lesssim \log \log L \,.$$

We now split the complement of B into the following three regions:

$$\Gamma_1 = \left\{ x = (x_1, x_2, a) \in B^c : a < a^*, \, |x| < f(a) \right\},$$

$$\Gamma_2 = \left\{ x = (x_1, x_2, a) \in B^c : a \geq a^* \right\},$$

$$\Gamma_3 = \left\{ x = (x_1, x_2, a) \in B^c : a < a^*, \, |x| \geq f(a) \right\},$$

where $a^* = \mathrm{e}^{-r_B/8}$ and $f(a) = \mathrm{e}^{\sqrt{\log a^{-1}}}$.

Step 3. We first estimate the integral of $\mathcal{M} f_L$ over the region Γ_1, and here we use the simple fact that $\mathcal{M} f_L \leq \mathcal{M} \chi_{R_L} + \mathcal{M} \chi_{R_0}$. For any point x in Γ_1 we have

$$\frac{a^{-1}(1 + |x|^2)}{2} < \frac{a + a^{-1} + a^{-1}|x|^2}{2} = \cosh r(x) < \mathrm{e}^{r(x)},$$

which implies that

$$\log a^{-1} \lesssim r(x).$$

From (4.3) we obtain

$$
\begin{aligned}
\int_{\Gamma_1} \mathcal{M} \chi_{R_0} \, \mathrm{d}\rho &\lesssim \int_0^{a^*} \frac{\mathrm{d}a}{a} \int_{|x| < f(a)} a^{-1} \left[\log a^{-1} \right]^{-2} \frac{\mathrm{d}x}{a^{-1}(1 + |x|^2)} \\
&\lesssim \int_0^{a^*} \frac{\mathrm{d}a}{a} \left[\log a^{-1} \right]^{-2} \log f(a) \\
&\lesssim \int_0^{a^*} \frac{\mathrm{d}a}{a} \left[\log a^{-1} \right]^{-3/2} \\
&\lesssim 1.
\end{aligned}
$$
(4.7)

With a translation argument as in (4.5), we see that

$$\int_{\Gamma_1} \mathcal{M} \chi_{R_L} \, \mathrm{d}\rho \leq \int_{\tau_L(\Gamma_1)} \mathcal{M} \chi_{R_0} \, \mathrm{d}\rho.$$
(4.8)

Now, any point (x_1, x_2, a) in $\tau_L(\Gamma_1)$ satisfies $a < a^*$ and $|x| < f(a) + L$, which implies $f(a) > f(a^*) = \mathrm{e}^{\sqrt{r_B/8}} = \mathrm{e}^{(\log L)^{\alpha/2}/\sqrt{8}} > L$, if L is large enough. Then $|x| < 2f(a)$, and the right-hand side of (4.8) can be estimated as in (4.7), with $f(a)$ replaced by $2f(a)$. We conclude that

$$\int_{\Gamma_1} \mathcal{M} \chi_{R_L} \, \mathrm{d}\rho \lesssim 1.$$
(4.9)

Step 4. In order to estimate the integrals over Γ_2 and Γ_3, we first write the convolution $f_L * p_t(x)$ at a point $x \in B^c$ as follows:

$$
\begin{aligned}
f_L * p_t(x) &= \int_{R_L} p_t\left(y^{-1}x\right) \mathrm{d}\lambda(y) - \int_{R_0} p_t\left(y^{-1}x\right) \mathrm{d}\lambda(y) \\
&= \int_{R_0} \left[p_t\left(y^{-1}(-L, 0, 1)x\right) - p_t\left(y^{-1}x\right) \right] \mathrm{d}\lambda(y).
\end{aligned}
$$

Let now $y^{-1} = (y_1, y_2, b)$ be any point in $(R_0)^{-1}$ and $x = (x_1, x_2, a)$ any point in B^c. Then $y^{-1}(-L, 0, 1)x = y^{-1}(-L, 0, 1)y \cdot y^{-1}x = (-bL, 0, 1)\, y^{-1}x$, and the Mean Value Theorem implies

$$
\begin{aligned}
p_t\left(y^{-1}(-L, 0, 1)x\right) - p_t\left(y^{-1}x\right) &= -bL\, \partial_1 p_t\left((s, 0, 1)\, y^{-1}x\right) \\
&= -bL\, \partial_1 p_t\left(s + y_1 + bx_1, y_2 + bx_2, ba\right),
\end{aligned}
$$

for some $s \in (-bL, 0)$. We now use the fact that $X_1 = a\partial_1$ and the explicit expression for the derivative $X_1 p_t$ given by Lemma 2.1 (i), to obtain

$$p_t\big(y^{-1}(-L, 0, 1)x\big) - p_t(y^{-1}x)$$

$$= -\frac{1}{8\pi^{3/2}}\, bL\, (ab)^{-1}\, t^{-3/2}\, \mathrm{e}^{-\frac{r^2}{4t}} \Big(-\frac{r^2}{2t \sinh r} + \frac{\sinh r - r \cosh r}{\sinh^2 r} \Big) \frac{s + y_1 + bx_1}{\sinh r}\, \frac{1}{ab}$$

$$= -\frac{1}{8\pi^{3/2}}\, \frac{L}{a^2 b}\, t^{-3/2}\, \mathrm{e}^{-\frac{r^2}{4t}} \Big(-\frac{r^2}{2t \sinh^2 r} + \frac{\sinh r - r \cosh r}{\sinh^3 r} \Big) (s + y_1 + bx_1),$$

where $r = r\big((s, 0, 1)\, y^{-1} x\big) \gtrsim 1$. Taking the supremum in t and applying (2.6), we deduce that

$$\tag{4.10} \sup_t \big|p_t\big(y^{-1}(-L, 0, 1)x\big) - p_t(y^{-1}x)\big| \lesssim \frac{L}{a^2 b}\, r^{-3}\, \frac{r}{\sinh^2 r}\, |s + y_1 + bx_1|$$

$$\lesssim \frac{L}{a^2}\, r^{-2}\, \frac{1}{\sinh^2 r}\, (L + |x|),$$

since $y^{-1} \in (R_0)^{-1}$, $x \in B^c$ and $s \in (-bL, 0)$. By the triangle inequality,

$$|r\big((s, 0, 1) \cdot y^{-1} x\big) - r(x)| = |d\big(x, y \cdot (-s, 0, 1)\big) - d(x, e)|$$

$$\leq d\big(y \cdot (-s, 0, 1), e\big)$$

$$\leq d\big(y \cdot (-s, 0, 1), y\big) + d\big(y, e\big)$$

$$= d\big((-s, 0, 1), e\big) + d\big(y, e\big)$$

$$\lesssim \log L,$$

where we also used (1.1). Since $r(x) > (\log L)^\alpha$, we get $r\big((s, 0, 1)\, y^{-1} x\big) \sim r(x)$ and

$$\sinh^{-2} r\big((s, 0, 1) \cdot y^{-1} x\big) \lesssim \sinh^{-2}\big[r(x) - \log L\big] \lesssim \frac{L^2}{\sinh^2 r(x)}\,.$$

This and (4.10) imply that

$$\sup_t \big|p_t\big(y^{-1}(-L, 0, 1)x\big) - p_t(y^{-1}x)\big| \lesssim \frac{L^3}{a^2}\, r(x)^{-2}\, \frac{1}{\sinh^2 r(x)}\, (L + |x|),$$

which allows us to conclude that

$$\tag{4.11} \sup_t |f_L * p_t(x)| \lesssim \frac{L^3}{a^2}\, r(x)^{-2}\, \frac{1}{\sinh^2 r(x)}\, (L + |x|).$$

Let us now estimate $\mathcal{M} f_L$ in the region Γ_2. Using (4.11) and the fact that $|x| \lesssim a^{1/2} \cosh^{1/2} r(x)$, we get

$$\mathcal{M} f_L(x) \lesssim \frac{L^4}{a^2}\, \frac{r(x)^{-2}}{\sinh^2 r(x)} + \frac{L^3}{a^{3/2}}\, \frac{r(x)^{-2}}{\sinh^{3/2} r(x)} \qquad \forall x \in \Gamma_2\,.$$

By applying (2.1), we then obtain

$$\tag{4.12}$$

$$\int_{\Gamma_2} \mathcal{M} f_L \, \mathrm{d}\rho$$

$$\lesssim (a^*)^{-1}\, L^4 \int_{r_B}^{\infty} \frac{r^{-2}}{\sinh^2 r}\, r \sinh r \, \mathrm{d}r + (a^*)^{-1/2}\, L^3 \int_{r_B}^{\infty} \frac{r^{-2}}{\sinh^{3/2} r}\, r \sinh r \, \mathrm{d}r$$

$$\lesssim (a^*)^{-1}\, L^4\, \mathrm{e}^{-r_B} + (a^*)^{-1/2} L^3\, \mathrm{e}^{-r_B/2}$$

$$\lesssim 1\,.$$

We finally estimate $\mathcal{M}f_L$ in the region Γ_3. For any point x in Γ_3 we have $\cosh r(x) \sim a^{-1}|x|^2$, and $r(x) \sim \log(a^{-1}|x|^2)$, and also $|x| > \mathrm{e}^{\sqrt{\log(1/a^*)}} = \mathrm{e}^{\sqrt{(\log L)^\alpha/8}} > L$ for large L. Thus (4.11) implies

$$\mathcal{M}f_L(x) \lesssim \frac{L^3}{a^2}\,[\log(a^{-1}|x|^2)]^{-2}\,\frac{|x|}{[a^{-1}|x|^2]^2} \qquad \forall x \in \Gamma_3\,.$$

We now integrate over the region Γ_3, making the change of variables $a^{-1/2}x = y$:

$$\int_{\Gamma_3} \mathcal{M}f_L\,\mathrm{d}\rho \lesssim L^3 \int_0^{a^*} \frac{\mathrm{d}a}{a^3} \int_{|x|>f(a)} [\log(a^{-1}|x|^2)]^{-2}\,\frac{|x|\,\mathrm{d}x}{[a^{-1}|x|^2]^2}$$

$$= L^3 \int_0^{a^*} \frac{\mathrm{d}a}{a^{3/2}} \int_{|y|>a^{-1/2}f(a)} [\log(|y|^2)]^{-2}\,\frac{|y|\,\mathrm{d}y}{|y|^4}$$

(4.13)
$$\lesssim L^3 \int_0^{a^*} \frac{\mathrm{d}a}{a^{3/2}}\,\frac{1}{[a^{-1/2}f(a)]\,[\log a^{-1} + \log f(a)^2]^2}$$

$$\lesssim L^3 \int_0^{a^*} \frac{\mathrm{d}a}{a}\,\frac{1}{f(a)\,[\log a^{-1}]^2}$$

$$= L^3 \int_0^{a^*} \frac{\mathrm{d}a}{a}\,\frac{1}{\mathrm{e}^{\sqrt{\log a^{-1}}}\,[\log a^{-1}]^2}\,.$$

An easy calculation shows that

$$\int_0^{a^*} \frac{\mathrm{d}a}{a}\,\frac{1}{\mathrm{e}^{\sqrt{\log a^{-1}}}\,[\log a^{-1}]^2} \lesssim \mathrm{e}^{-\sqrt{\log(a^*)^{-1}}}\,\big[\log(a^*)^{-1}\big]^{-3/2}\,,$$

and $\log(a^*)^{-1} = (\log L)^\alpha/8$. Thus by (4.13) we get

(4.14)
$$\int_{\Gamma_3} \mathcal{M}f_L\,\mathrm{d}\rho \lesssim 1\,,$$

since $\alpha > 2$.

Summing up the estimates (4.1), (4.6), (4.7), (4.9), (4.12), (4.14), one concludes that

$$\int \mathcal{M}f_L\,\mathrm{d}\rho \lesssim \log\log L\,,$$

and Proposition (4.4) is proved.   □

By Proposition 4.4 and Lemma 4.2 it follows that

$$\lim_{L \to +\infty} \frac{\|f_L\|_{H^1}}{\|\mathcal{M}f_L\|_1} = +\infty\,.$$

Thus there is no converse estimate to that of Theorem 1.3 (i).

We can now finish the proof of Theorem 1.3 (ii). Define the function f by

$$f(x_1, x_2, a) = \frac{1}{x_1(\log x_1)^{3/2}}$$

in the set $\{(x_1, x_2, a) : x_1 > 3,\ |x_2| < 1,\ \mathrm{e}^{-1} < a < \mathrm{e}\}$, $f = c_0$ in R_0 and $f = 0$ elsewhere, with the constant c_0 chosen so that $\int f\,\mathrm{d}\rho = 0$.

First we show that f does not lie in H^1. We truncate the BMO function g from the proof of Lemma 4.2, and let for $N = 1, 2, \dots$

$$g_N = \min\big(\max(g, -N), N\big)\,.$$

These L^∞ functions satisfy $\|g_N\|_{BMO} \leq \|g\|_{BMO}$. It is easy to see that

$$\lim_{N \to \infty} \int g_N f \, d\rho = +\infty \,. \tag{4.15}$$

If f were in H^1 and thus had an atomic decomposition converging in L^1, we would get $\langle g_N, f \rangle = \int g_N f \, d\rho$. This would be a contradiction, since the g_N define uniformly bounded functionals on H^1.

To estimate the L^1 norm of $\mathcal{M}f$, we approximate f with a linear combination of the functions f_L with even, integer values of L. Let

$$\tilde{f} = \sum_{k=2}^{\infty} c_k f_{2k} \,,$$

where c_k is the mean value of f in R_{2k}. For every (x_1, x_2, a) in R_{2k}

$$\left| f(x_1, x_2, a) - c_k \right| \lesssim x_1^{-2} (\log x_1)^{-3/2} \lesssim k^{-2} \,.$$

This means that the restriction of the difference $f - \tilde{f}$ to R_{2k} is k^{-2} times an atom. Observe also that f and $\tilde{f}$ coincide in R_0.

From Proposition 4.4 and Theorem 1.3 (i), we conclude

$$\|\mathcal{M}f\|_1 \leq \|\mathcal{M}\tilde{f}\|_1 + \|\mathcal{M}(f - \tilde{f})\|_1 \lesssim \sum_{k=2}^{\infty} c_k \log \log k + \sum_{k=2}^{\infty} k^{-2}.$$

Since $c_k \lesssim k^{-1} (\log k)^{-3/2}$, these quantities are finite. Theorem 1.3 is completely proved.

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